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Abstract: Using Malliavin weights in a jump-diffusion model we obtain an expression for Theta (the sensitivity of an option price with respect to the time remaining until exercise), with application to European and Asian options with non-smooth payoff function. In time inhomogeneous models our formula applies to the derivative with respect to the maturity date T, and its proof can be viewed as a generalization of Dupire's integration by parts to arbitrary payoff functions. In the time homogeneous case, our result applies to the derivative with respect to the current date t, but our representation formula differs from the one obtained from the Black-Scholes PDE in terms of Delta and Gamma. Optimal weights are computed by minimization of variance and numerical simulations are presented.

Key-words: Greeks, Theta, sensitivity analysis, jump-diffusion models, Malliavin calculus.

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Calcul numérique de Theta par intégration par parties dans un modèle de diffusion avec sauts

Résumé : Par le calcul de Malliavin nous obtenons une expression de Theta (la sensibilité d'un prix d'option par rapport au temps restant jusqu'à l'exercice), avec application aux options européennes et asiatiques à fonction de payoff irrégulière. Dans le cas non-homogène en temps, notre formule s'applique à la dérivée par rapport à la date d'exercice T, et sa preuve peut être vue comme une généralisation de l'argument de Dupire à des fonctions de payoff arbitraires. Dans le cas homogène en temps, notre résultat s'applique à la dérivée par rapport à la date courante t, mais notre formule de représentation diffère de celle obtenue par l'EDP de Black-Scholes en termes de Delta et Gamma. Nous déterminons également des poids de variance minimale, et des simulations numériques sont présentées.

Mots-clés : Grecs, Theta, analyse de sensibilité, diffusions avec sauts, calcul de Malliavin.

1 Introduction

Predicting the variations of option prices can be a difficult task, and in order to make such predictions it is important to understand which factors contribute to the movement of prices, and with what effect. The Delta, Gamma, Vega, Rho and Theta of an option position, collectively known as the "Greeks", provide a way to measure the sensitivity of an option prices to factors such as spot price, volatility, interest rate and time respectively.

Sensitivity analysis using the Malliavin calculus in finance has been developed by several authors, starting with [7]. In this paper we aim at applying similar methods to the computation of Theta, which measures the variations of an option price with respect to the time until exercise, while other parameters remain constant. Our argument consists in a combination of Itô calculus with integration by parts on the Wiener space. Let

$$C(x,t,T) = e^{-\int_t^T r_s ds}\mathbb{E}\left[\phi(S_T)\Big|S_t = x\right]$$

denote the price at time t of a European type option with spot price x, interest rate and volatility functions r_t, $\sigma_t(\cdot)$, maturity T, strike price K, payoff function ϕ, and price process $(S_t)_{t\in[0,T]}$. In this general setting, two versions of Theta can be computed, namely

$$\text{Theta}_t = \frac{\partial C}{\partial t}(x,t,T), \quad \text{and} \quad \text{Theta}_T = \frac{\partial C}{\partial T}(x,t,T).$$

Theta_t is used for European options for which T is a fixed date, whereas Theta_T can be used in case T is a free parameter, e.g. for the choice of the exercise date of a European option, or for American type contracts.

As is well-known, Theta_t is involved in the Black-Scholes PDE: for example, in a continuous market with interest rate and volatility functions r_t, $\sigma_t(\cdot)$ we have

$$\text{Theta}_t = r_t C(x,t,T) - x r_t \frac{\partial C}{\partial x}(x,t,T) - \frac{1}{2}x^2\sigma_t^2(x)\frac{\partial^2 C}{\partial x^2}(x,t,T). \tag{1.1}$$

In case the Markov process $(S_t)_{t\in[0,T]}$ is time homogeneous, i.e. when the coefficients r, $\sigma(\cdot)$ are time independent, the price C satisfies

$$C(x,t,T) = e^{-(T-t)r}\mathbb{E}\left[\phi(S_{T-t})\right]$$

and becomes a function of the remaining time $\tau := T - t$ until exercise. In this case we have

$$\mathrm{Theta}_T = -\mathrm{Theta}_t = \frac{\partial C}{\partial \tau}(x, t, t + \tau),$$

which will be simply denoted by Theta.

In this paper we compute Theta_T in a time inhomogeneous setting, using the Itô formula. In the homogeneous case this yields a representation formula which differs from the one given for $\mathrm{Theta}_t = -\mathrm{Theta}_T$ by the Black-Scholes PDE (1.1) in terms of the Greeks Delta $\frac{\partial C}{\partial x}$ and Gamma $\frac{\partial^2 C}{\partial x^2}$, and relies on a single weight whose variance is minimized individually. Since our approach is directly based on time derivatives, it does not require the use of the first and second variation processes which would be normally needed in the computation of Gamma by integration by parts. In the particular case of European options in a continuous market, with price

$$C(x, t, T, K) = e^{-\int_t^T r_s ds}\mathbb{E}[(S_T - K)^+ \mid S_t = x],$$

our method extends the argument of [6] which yields the Dupire PDE

$$\mathrm{Theta}_T = -K r_T \frac{\partial C}{\partial K}(x, t, T, K) + \frac{1}{2}K^2 \sigma_T^2(K)\frac{\partial^2 C}{\partial K^2}(x, t, T, K).$$

We also apply our method to Asian options.

We proceed as follows. In Section 2 we recall some notation on the Malliavin calculus on the Wiener space and on stochastic calculus for jump processes. In Section 3 we review the computation of Theta_t via the Black-Scholes PDE in the jump-diffusion case using the Greeks Delta and Gamma, and the computation of Theta_T via the Dupire PDE for European options in a continuous market. We note that those representations formulas differ in general from the one obtained in this paper. In Section 4, using the Malliavin calculus, we obtain an expression of Theta_T in a jump-diffusion model which, in the time homogeneous case, reads

$$\mathrm{Theta} = e^{-\tau r}\mathbb{E}\left[\Lambda_\tau(u, v, w)\phi(S_\tau) + \int_{-\infty}^{+\infty}(\phi(S_\tau + c(S_\tau)y) - \phi(S_\tau))\mu(dy)\right],$$

with arbitrary payoff function $\phi(\cdot)$ and $S_0 = x$. Here, $\Lambda_\tau(u, v, w)$ is a random weight which is explicitly computed, $\mu(dx)$ and $c(\cdot)$ are respectively the (finite) Lévy measure and the volatility coefficient of the compound Poisson process driving $(S_t)_{t\in\mathbb{R}_+}$. As in

[4], [5], differentiation is performed only with respect to the Brownian component of the price process. The above expression is independent of the functional parameters u, v, w, and in Section 5 we determine the parameters yielding the best numerical performance by minimization of the variance of the weight $\Lambda_\tau(u, v, w)$, and find that this minimum is attained when u, v, w are constant functions. Asian options are considered in Section 6. Simulations for digital and European options, using the Monte Carlo method, are presented in Section 7 to compare the performance of the finite difference method to that of the Malliavin calculus approach. A localization approach is presented in Section 8 to reduce the singularity points phenomenon on the simulation graphs.

2 Notation

In this section we recall some facts and notation on the Malliavin calculus on the Wiener space, cf e.g. [10], and on stochastic calculus with jumps, see e.g. and [3] for a recent introduction with references. We work on a product

$$(\Omega, P) = (\Omega_W \times \Omega_X, P_W \otimes P_X)$$

of probability spaces on which are respectively defined a standard Brownian motion $(W_t)_{t\in\mathbb{R}_+}$ and a compound Poisson process $(X_t)_{t\in\mathbb{R}_+}$ independent of $(W_t)_{t\in\mathbb{R}_+}$, with Lévy measure $\mu(dy)$ and finite intensity

$$\lambda = \int_{-\infty}^{\infty} y\mu(dy),$$

which can be represented as

$$X_t = \sum_{k=1}^{N_t} U_k, \qquad t \in \mathbb{R}_+, \tag{2.1}$$

where $(N_t)_{t\in\mathbb{R}_+}$ is a Poisson process with intensity λ and $(U_k)_{k\geq 1}$ is an i.i.d. sequence of random variables with probability distribution $\nu(dx) = \lambda^{-1}\mu(dx)$. We denote by $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$ the filtration generated by $(W_t, X_t)_{t\in\mathbb{R}_+}$.

We consider gradient and divergence operators D and δ acting on the continuous component of jump-diffusion random functionals. Let $D : L^2(\Omega) \to L^2(\Omega \times \mathbb{R}_+)$ denote the (unbounded) Malliavin gradient on the Wiener space, i.e.

$$D_t F(\omega_W, \omega_X) = \sum_{k=1}^{n} 1_{[0,t_k]}(t)\partial_k f(W_{t_1}, \dots, W_{t_n}, \omega_X)$$

for F a random variable of the form

$$F(\omega_W, \omega_X) = f(W_{t_1}, \dots, W_{t_n}, \omega_X),$$

where $f(\cdot, \omega_X) \in \mathcal{C}_b^\infty(\mathbb{R}^n)$, $P_X(d\omega_X)$-a.s., is uniformly bounded on Ω_X. Denote by $\langle \cdot, \cdot \rangle_{L^2(\mathbb{R}_+)}$ and $\| \cdot \|$ the scalar product and norm in $L^2(\mathbb{R}_+)$, and define

$$D_u F = \langle u, DF \rangle, \quad u \in L^2(\Omega \times \mathbb{R}_+).$$

Let

$$I_n(f_n)(\omega_W, \omega_X) = n! \int_0^\infty \cdots \int_0^{t_2} f_n(t_1, \dots, t_n, \omega_X) dW_{t_1} \cdots dW_{t_n}$$

denote the multiple stochastic integral of the symmetric function $f_n \in L^2(\mathbb{R}_+^n \times \Omega_X)$ with respect to Brownian motion $(W_t)_{t \in \mathbb{R}_+}$. Recall that we have

$$D_t I_n(f_n) = n I_{n-1}(f_n(\cdot, t, \omega_X)), \qquad t \in \mathbb{R}_+,$$

and the isometry formula

$$\mathbb{E}[I_n(f_n) I_m(g_m)] = n! 1_{\{n=m\}} \mathbb{E}[\langle f_n, g_m \rangle_{L^2(\mathbb{R}_+^n)}].$$

Any $F \in L^2(\Omega_W \times \Omega_X) \simeq L^2(\Omega_W; L^2(\Omega_X))$ admits a chaos decomposition of the form

$$F = E[F] + \sum_{n=1}^\infty I_n(f_n), \tag{2.2}$$

where $f_n \in L^2(\mathbb{R}_+^n \times \Omega_X)$, $n \geq 1$, and the domain $\mathrm{Dom}\,(D)$ of D consists in the set of functionals F written as (2.2) and satisfying

$$\mathbb{E}\left[\sum_{n=1}^\infty n! n \| f_n \|_{L^2(\mathbb{R}_+^n)}^2 \right] < \infty.$$

In the sequel we will generally drop the indices ω_W, ω_X. The (unbounded) divergence operator $\delta : L^2(\Omega \times \mathbb{R}_+) \to L^2(\Omega)$ adjoint of D, also called the Skorohod integral, satisfies the duality relation

$$\mathbb{E}[\langle DF, u \rangle] = \mathbb{E}[F \delta(u)], \qquad F \in \mathrm{Dom}\,(D), \quad u \in \mathrm{Dom}\,(\delta),$$

and the divergence formula

$$\delta(uF) = F\delta(u) - D_u F, \qquad u \in \mathrm{Dom}\,(\delta), \quad F \in \mathrm{Dom}\,(D), \tag{2.3}$$

which shows that $uF \in \mathrm{Dom}\,(\delta)$ provided $uF \in L^2(\Omega \times \mathbb{R}_+)$ and the right-hand side belongs to $L^2(\Omega)$. Recall also that δ coincides with Itô's stochastic integral on square-integrable adapted processes, in particular

$$\delta(u) = \int_0^\infty u_t dW_t$$

for all adapted and square-integrable process $(u_t)_{t\in\mathbb{R}_+}$, and

$$\delta(u) = I_1(u), \qquad u \in L^2(\mathbb{R}_+).$$

In the sequel we will consider a Markovian jump-diffusion price process $(S_t)_{t\in\mathbb{R}_+}$ given by

$$\begin{cases} dS_t = a_t(S_t)dt + b_t(S_t)dW_t + c_t(S_{t^-})dX_t, \\ \\ S_0 = x, \end{cases}$$

where $a_t(\cdot)$, $b_t(\cdot)$, $c_t(\cdot)$ are $\mathcal{C}^1$ Lipschitz functions, uniformly in $t \in [0,T]$, $T > 0$. In the case of a time homogeneous geometric model under the risk neutral measure, the coefficients $a(\cdot)$, $b(\cdot)$ and $c(\cdot)$ will be given by

$$\begin{cases} a(y) = y(r - \lambda\eta(y)), \\ \\ b(y) = y\sigma(y), \\ \\ c(y) = y\eta(y), \end{cases}$$

where r and $\sigma(\cdot)$, $\eta(\cdot)$ represent the interest rate, and the continuous and jump volatility functions.

Itô's formula for $(S_t)_{t\in\mathbb{R}_+}$ reads

$$\begin{aligned} \phi(S_t) &= \phi(S_s) + \int_s^t \phi'(S_u)a_u(S_u)du + \int_s^t \phi'(S_u)b_u(S_u)dW_u \\ &\quad + \frac{1}{2}\int_s^t \phi''(S_u)b_u^2(S_u)du + \sum_{s<u\le t} \left(\phi(S_{u^-} + c_u(S_{u^-})\Delta X_u) - \phi(S_{u^-})\right), \end{aligned}$$

$0 \le s \le t$, and yields

$$\mathbb{E}[\phi(S_t)] = \mathbb{E}[\phi(S_s)] + \mathbb{E}\left[\int_s^t \phi'(S_u)a_u(S_u)du\right] + \frac{1}{2}\mathbb{E}\left[\int_s^t \phi''(S_u)b_u^2(S_u)du\right]$$

$$+\lambda \mathbb{E}\left[\int_s^t \int_{-\infty}^{+\infty} (\phi(S_u + zc_u(S_u)) - \phi(S_u))\nu(dz)du\right]. \quad (2.4)$$

For example, if

$$X_t = a_1 N_t^1 + \cdots + a_d N_t^d, \qquad t \in \mathbb{R}_+, \quad (2.5)$$

where $(N_t^k)_{t\in\mathbb{R}_+}$, $k = 1, \ldots, d$, are independent Poisson processes with respective intensities $\lambda_1, \ldots, \lambda_d$, we have $\lambda = \lambda_1 + \cdots + \lambda_d$ and

$$\nu(dx) = \frac{\lambda_1}{\lambda}\delta_{a_1}(dx) + \cdots + \frac{\lambda_d}{\lambda}\delta_{a_d}(dx),$$

and

$$\begin{aligned}\mathbb{E}[\phi(S_t)] &= \mathbb{E}[\phi(S_s)] + \mathbb{E}\left[\int_s^t \phi'(S_u)a_u(S_u)du\right] + \frac{1}{2}\mathbb{E}\left[\int_s^t \phi''(S_u)b_u^2(S_u)du\right] \\ &\quad + \sum_{k=1}^d \lambda_k \mathbb{E}\left[\int_s^t (\phi(S_u + a_k c_u(S_u)) - \phi(S_u))du\right], \qquad 0 \le s \le t.\end{aligned}$$

In the linear case we let

$$\begin{cases} a(y) = (r - \lambda\eta)y, \\ b(y) = \sigma y, \\ c(y) = \eta y, \end{cases}$$

and get

$$\begin{cases} dS_s = rS_s ds + \sigma S_s dW_s + \eta S_{s^-}(dX_s - \lambda ds), \\ S_0 = x, \end{cases} \quad (2.6)$$

with solution

$$S_t = x \exp\left(\left(r - \lambda\eta - \frac{\sigma^2}{2}\right)t + \sigma W_t\right) \prod_{0<s\le t} (1 + \eta\Delta X_s), \qquad t \in \mathbb{R}_+. \quad (2.7)$$

If $(X_t)_{t\in\mathbb{R}_+}$ has the form (2.5) we obtain

$$S_t = x \exp\left(\left(r - \lambda\eta - \frac{\sigma^2}{2}\right)t + \sigma W_t\right)(1 + \eta a_1)^{N_t^1} \cdots (1 + \eta a_d)^{N_t^d}, \qquad t \in \mathbb{R}_+.$$

3 PDE approaches to Theta

In this section we review the computation of Theta_t and Theta_T in particular cases, using respectively the Black-Scholes and Dupire PDEs.

Consider an option with payoff function ϕ and price

$$C(x,t,T) = e^{-\int_t^T r_s ds}\mathbb{E}\left[\phi(S_T)\Big|S_t = x\right].$$

Since $t \mapsto e^{\int_t^T r_s ds} C(S_t,t,T)$ is a martingale, from (2.4), $C(x,t,T)$ satisfies the Black-Scholes PDE

$$\begin{aligned}\text{Theta}_t &= \frac{\partial C}{\partial t}(x,t,T) \qquad (3.1)\\ &= r_t C(x,t,T) - a_t(x)\frac{\partial C}{\partial x}(x,t,T) - \frac{1}{2}b_t^2(x)\frac{\partial^2 C}{\partial x^2}(x,t,T)\\ &\quad -\lambda\int_{-\infty}^{+\infty}(C(x+zc_t(x),t,T)) - C(x,t,T))\nu(dz),\end{aligned}$$

with

$$\text{Delta} = \frac{\partial C}{\partial x}(x,t,T) = e^{-\int_t^T r_s ds}\mathbb{E}\left[Y_T\phi'(S_T)\Big|S_t = x\right], \qquad (3.2)$$

where $(Y_t)_{t\in\mathbb{R}_+} = (\partial_x S_t)_{t\in\mathbb{R}_+}$ is the first variation process solution of

$$\begin{cases} dY_t = a_t'(S_t)Y_t dt + b_t'(S_t)Y_t dW_t + c_t'(S_{t^-})Y_{t^-}dX_t, \\ Y_0 = 1, \end{cases}$$

and

$$\text{Gamma} = \frac{\partial^2 C}{\partial x^2}(x,t,T) = e^{-\int_t^T r_s ds}\mathbb{E}\left[Z_T\phi'(S_T) + (Y_T)^2\phi''(S_T)\Big|S_t = x\right], \qquad (3.3)$$

where $(Z_t)_{t\in\mathbb{R}_+} = (\partial_x^2 S_t)_{t\in\mathbb{R}_+}$ is the second variation process. Here, $a_t'(z)$, $b_t'(z)$ and $c_t'(z)$ denote the partial derivatives of these functions with respect to z.

The first and second derivatives on ϕ in the expressions (3.2), (3.3) of Delta and Gamma can be removed via integration by parts and the Malliavin calculus, as in e.g. [5], [4], to yield an expression for Theta_t. However the computation of Gamma directly involves the first and second variation processes Y_t and Z_t. Namely in [7], the expression (3.2) is rewritten as

$$\text{Delta} = \frac{\partial C}{\partial x}(x,t,T) = \frac{e^{-\int_t^T r_s ds}}{T-t}\mathbb{E}\left[\phi(S_T)\int_t^T \frac{Y_s}{b_s(S_s)}dW_s \Big| S_t = x\right], \tag{3.4}$$

using the expression of the Malliavin derivative of $S_T \in \mathrm{Dom}\,(D)$ in terms of the first variation process as:

$$\frac{Y_s}{b_s(S_s)}D_s\phi(S_T) = Y_T\phi'(S_T), \qquad 0 \le s \le \tau,\ a.s., \tag{3.5}$$

cf. [10]. In the next section we present a computation of Theta_T by integration by parts which, in the time homogeneous case, yields a different representation for $\text{Theta}_t = -\text{Theta}_T$. It does not directly use the first and second variation processes, and involves only elementary Wiener integrals of deterministic functions instead of Itô stochastic integrals of adapted processes as in (3.5).

On the other hand, in the case of European options in a continuous market, i.e. with $c_t(\cdot) = 0$, $a_t(y) = \alpha_t y$, with payoff function $\phi(x) = (x-K)^+$, and price

$$C(x,t,T,K) = e^{-\int_t^T r_s ds}\mathbb{E}[(S_T - K)^+ \mid S_t = x],$$

Dupire's formula [6] reads

$$b_T(K) = \sqrt{2\frac{(r_T - \alpha_T)C + \dfrac{\partial C}{\partial T} + K r_T \dfrac{\partial C}{\partial K}}{K^2 \dfrac{\partial^2 C}{\partial K^2}}}$$

where $\frac{\partial C}{\partial T}(x,t,T,K)$ coincides with Theta_T. In other terms,

$$\begin{aligned}\text{Theta}_T &= \frac{\partial C}{\partial T}(x,t,T,K) \\ &= (\alpha_T - r_T)C(x,t,T,K) + \frac{K^2 b_T^2(K)}{2}\frac{\partial^2 C}{\partial K^2}(x,t,T,K) - K\alpha_T\frac{\partial C}{\partial K}(x,t,T,K),\end{aligned} \tag{3.6}$$

where $-\frac{\partial^2 C}{\partial y^2}(x,t,T,y)$ coincides with the density function $dP(S_T = y \mid S_t = x)/dy$. Relation (3.6) can be proved by application of (2.4) on $[0,T]$, differentiation with respect to T, and integration by parts with respect to dy on $\mathbb{R}$. The computation of Theta$_T$ presented in the next section follows the same steps, replacing integration by parts on $\mathbb{R}$ with the duality formula on the Wiener space. As such it can be viewed as a generalization of Dupire's argument to arbitrary payoff functions in a jump diffusion market.

4 Computation of Theta$_T$

Consider an option with payoff function ϕ and price

$$C(x,t,T) = e^{-\int_t^T r_s ds} \mathbb{E}\left[\phi(S_T) \Big| S_t = x \right].$$

Theta$_T$ can be approximated by finite differences as

$$\text{Theta}_T = \frac{C(x,t,(1+\varepsilon)T) - C(x,t,(1-\varepsilon)T)}{2T\varepsilon}. \tag{4.1}$$

Alternatively, the derivative with respect to T can be put inside the expectation if ϕ is differentiable.

Consider $(S^x_{t,s})_{s\in[t,\infty)}$ given by the jump-diffusion equation

$$\begin{cases} dS^x_{t,s} = a_s(S^x_{t,s})ds + b_s(S^x_{t,s})dW_s + c_s(S^x_{t,s^-})dX_s, \\ \\ S^x_{t,t} = x. \end{cases} \tag{4.2}$$

Using Itô's formula and (2.4) we have:

$$\begin{aligned} C(x,t,T) &= e^{-\int_t^T r_s ds} \mathbb{E}\left[\phi(S^x_{t,T})\right] \\ &= \phi(x) - \mathbb{E}\left[\int_t^T r_s e^{-\int_t^s r_p dp} \phi(S^x_{t,s}) ds \right] \\ &\quad + \mathbb{E}\left[\int_t^T e^{-\int_t^s r_p dp} \phi'(S^x_{t,s}) a_s(S^x_{t,s}) ds \right] + \mathbb{E}\left[\int_t^T e^{-\int_t^s r_p dp} \phi'(S^x_{t,s}) b_s(S^x_{t,s}) dW_s \right] \\ &\quad + \frac{1}{2}\mathbb{E}\left[\int_t^T e^{-\int_t^s r_p dp} \phi''(S^x_{t,s}) b^2_s(S^x_{t,s}) ds \right] \\ &\quad + \lambda \mathbb{E}\left[\int_t^T e^{-\int_t^s r_p dp} \int_{-\infty}^{+\infty} (\phi(S^x_{t,s} + z c_s(S^x_{t,s})) - \phi(S^x_{t,s})) \nu(dz) ds \right] \end{aligned}$$

$$
\begin{aligned}
&= \phi(x) - \int_t^T r_s e^{-\int_t^s r_p dp} \mathbb{E}\left[\phi(S^x_{t,s})\right] ds + \int_t^T e^{-\int_t^s r_p dp} \mathbb{E}\left[\phi'(S^x_{t,s}) a_s(S^x_{t,s})\right] ds \\
&\quad + \frac{1}{2} \int_t^T e^{-\int_t^s r_p dp} \mathbb{E}\left[\phi''(S^x_{t,s}) b_s^2(S^x_{t,s})\right] ds \\
&\quad + \lambda \int_t^T e^{-\int_t^s r_p dp} \mathbb{E}\left[\int_{-\infty}^{+\infty} (\phi(S^x_{t,s} + z c_s(S^x_{t,s})) - \phi(S^x_{t,s}))\nu(dz)\right] ds,
\end{aligned}
$$

hence Theta_T can be expressed as

$$
\begin{aligned}
\mathrm{Theta}_T &= \frac{\partial}{\partial T}\left(e^{-\int_t^T r_s ds} \mathbb{E}\left[\phi(S^x_{t,T})\right]\right) \qquad (4.3) \\
&= -r_T e^{-\int_t^T r_p dp} \mathbb{E}\left[\phi(S^x_{t,T})\right] + e^{-\int_t^T r_p dp} \mathbb{E}\left[\phi'(S^x_{t,T}) a_T(S^x_{t,T})\right] \\
&\quad + \frac{1}{2} e^{-\int_t^T r_p dp} \mathbb{E}\left[\phi''(S^x_{t,T}) b_T^2(S^x_{t,T})\right] \\
&\quad + \lambda e^{-\int_t^T r_s ds} \mathbb{E}\left[\int_{-\infty}^{+\infty} (\phi(S^x_{t,T} + z c_T(S^x_{t,T})) - \phi(S^x_{t,T}))\nu(dz)\right],
\end{aligned}
$$

which differs from (3.1), including in the time-homogeneous case. Note that unlike the PDE approach of Section 3, this method does not seem to be applicable to the computation of Theta_t in a time-inhomogeneous setting. The above expression fails when ϕ is not twice differentiable. The aim of Proposition 4.1 below is to present a Malliavin type formula for Theta_T, which avoids the use of finite differences and does not require any smoothness on ϕ. This will allow us in particular to consider non-smooth payoff functions, as e.g. in the case of digital options. The derivatives on ϕ will be removed by integration by parts on the Wiener space, using the expression

$$
\phi'(S^x_{t,T}) = \frac{D_u \phi(S^x_{t,T})}{D_u S^x_{t,T}}, \qquad u \in L^2([t,T]). \qquad (4.4)
$$

The jump component of the above formula is left untouched since it does not contain any derivative.

Given $u, v, w \in L^2([t,T])$ such that $D_u S^x_{t,T}$, $D_v S^x_{t,T}$, $D_w S^x_{t,T}$ are a.s. non-zero, let the weight $\Lambda_{t,T}(u,v,w)$ be defined by

$$
\Lambda_{t,T}(u,v,w) = -a'_T(S^x_{t,T}) - r_T + a_T(S^x_{t,T})\left(\frac{I_1(u)}{D_u S^x_{t,T}} + \frac{D_u^2 S^x_{t,T}}{|D_u S^x_{t,T}|^2}\right)
$$

$$+\frac{1}{2}\left(\left(\frac{b_T^2(S_{t,T}^x)}{D_vS_{t,T}^x}I_1(v)-2b_T(S_{t,T}^x)b_T'(S_{t,T}^x)\ +\frac{b_T^2(S_{t,T}^x)D_v^2S_{t,T}^x}{|D_vS_{t,T}^x|^2}\right)\left(\frac{I_1(w)}{D_wS_{t,T}^x}+\frac{D_w^2S_{t,T}^x}{|D_wS_{t,T}^x|^2}\right)\right.$$
$$+\frac{b_T^2(S_{t,T}^x)}{D_wS_{t,T}^xD_vS_{t,T}^x}\left(I_1(v)\frac{D_wD_vS_{t,T}^x}{D_vS_{t,T}^x}-\langle v,w\rangle-\frac{D_wD_v^2S_{t,T}^x}{D_vS_{t,T}^x}\ +2\frac{D_wD_vS_{t,T}^xD_v^2S_{t,T}^x}{|D_vS_{t,T}^x|^2}\right)$$
$$\left.-\frac{2b_T'(S_{t,T}^x)b_T(S_{t,T}^x)}{D_vS_{t,T}^x}\left(I_1(v)+\frac{D_v^2S_{t,T}^x}{D_vS_{t,T}^x}\right)\right)+b_T'(S_{t,T}^x)^2+b_T''(S_{t,T}^x)b_T(S_{t,T}^x).$$

Recall that the Wiener integrals $I_1(u)$, $I_1(v)$ and $I_1(w)$ in the above formula are centered Gaussian random variables.

Proposition 4.1 *Let $u,v,w\in L^2([t,T])$ such that $D_uS_{t,T}^x$, $D_vS_{t,T}^x$, $D_wS_{t,T}^x$ are a.s. non-zero, assume that $\Lambda_{t,T}(u,v,w)\in L^2(\Omega)$, and let $\phi:\mathbb{R}\to\mathbb{R}$ be a Lipschitz function. We have*

$$\mathrm{Theta}_T=$$
$$e^{-\int_t^T r_sds}\mathbb{E}\left[\Lambda_{t,T}(u,v,w)\phi(S_{t,T}^x)+\lambda\int_{-\infty}^{+\infty}(\phi(S_{t,T}^x+zc_T(S_{t,T}^x))-\phi(S_{t,T}^x))\nu(dz)\right].$$

Proof. Using (2.3) and (4.4) we get for $u\in L^2([t,T])$ and ϕ, g_T, sufficiently smooth:

$$\begin{aligned}\mathbb{E}\left[\phi'(S_{t,T}^x)g_T(S_{t,T}^x)\right]&=\mathbb{E}\left[\frac{g_T(S_{t,T}^x)}{D_uS_{t,T}^x}D_u\phi(S_{t,T}^x)\right]\\
&=\mathbb{E}\left[\left\langle D\phi(S_{t,T}^x),\frac{g_T(S_{t,T}^x)}{D_uS_{t,T}^x}u\right\rangle\right]\\
&=\mathbb{E}\left[\phi(S_{t,T}^x)\delta\left(\frac{g_T(S_{t,T}^x)}{D_uS_{t,T}^x}u\right)\right]\\
&=\mathbb{E}\left[\phi(S_{t,T}^x)\left(\frac{g_T(S_{t,T}^x)}{D_uS_{t,T}^x}I_1(u)-D_u\left(\frac{g_T(S_{t,T}^x)}{D_uS_{t,T}^x}\right)\right)\right]\\
&=\mathbb{E}\left[\phi(S_{t,T}^x)\left(\frac{g_T(S_{t,T}^x)}{D_uS_{t,T}^x}I_1(u)-g_T'(S_{t,T}^x)+\frac{g_T(S_{t,T}^x)D_u^2S_{t,T}^x}{|D_uS_{t,T}^x|^2}\right)\right].\end{aligned}$$

With $g_T(\cdot)=a_T(\cdot)$ we obtain

$$\mathbb{E}\left[\phi'(S_{t,T}^x)a_T(S_{t,T}^x)\right]=\mathbb{E}\left[\phi(S_{t,T}^x)\left(\frac{a_T(S_{t,T}^x)}{D_uS_{t,T}^x}I_1(u)-a_T'(S_{t,T}^x)+\frac{a_T(S_{t,T}^x)D_u^2S_{t,T}^x}{|D_uS_{t,T}^x|^2}\right)\right],\tag{4.5}$$

while $g_T(\cdot) = b_T^2(\cdot)$ yields

$$\mathbb{E}\left[\phi''(S_{t,T}^x)b_T^2(S_{t,T}^x)\right] = \mathbb{E}\left[\phi'(S_{t,T}^x)\Gamma_{t,T}(v)\right],$$

where

$$\Gamma_{t,T}(v) = \frac{b_T^2(S_{t,T}^x)}{D_v S_{t,T}^x} I_1(v) - 2b_T(S_{t,T}^x)b_T'(S_{t,T}^x) + \frac{b_T^2(S_{t,T}^x)D_v^2 S_{t,T}^x}{|D_v S_{t,T}^x|^2}.$$

By a similar argument we get

$$\begin{aligned}
&\mathbb{E}\left[\phi''(S_{t,s}^x)b_s^2(S_{t,s}^x)\right] = \mathbb{E}\left[\phi'(S_{t,T}^x)\Gamma_{t,T}(v)\right] \\
&= \mathbb{E}\left[\Gamma_{t,T}(v)\frac{D_w\phi(S_{t,T}^x)}{D_w(S_{t,T}^x)}\right] \\
&= \mathbb{E}\left[\phi(S_{t,T}^x)\delta\left(w\frac{\Gamma_{t,T}(v)}{D_w S_{t,T}^x}\right)\right] \\
&= \mathbb{E}\left[\phi(S_{t,T}^x)\left(\frac{\Gamma_{t,T}(v)}{D_w S_{t,T}^x}I_1(w) - D_w\left(\frac{\Gamma_{t,T}(v)}{D_w S_{t,T}^x}\right)\right)\right] \\
&= \mathbb{E}\left[\phi(S_{t,T}^x)\left(\frac{\Gamma_{t,T}(v)}{D_w S_{t,T}^x}\left(I_1(w) + \frac{D_w^2 S_{t,T}^x}{D_w S_{t,T}^x}\right) - \frac{2b_T'(S_{t,T}^x)b_T(S_{t,T}^x)}{D_v S_{t,T}^x}\left(I_1(v) + \frac{D_v^2 S_{t,T}^x}{D_v S_{t,T}^x}\right)\right.\right. \\
&\quad + \frac{b_T^2(S_{t,T}^x)}{D_w S_{t,T}^x D_v S_{t,T}^x}\left(\frac{I_1(v)D_w D_v S_{t,T}^x}{D_v S_{t,T}^x} - \langle v, w\rangle - \frac{D_w D_v^2 S_{t,T}^x}{D_v S_{t,T}^x} + \frac{2D_w D_v S_{t,T}^x D_v^2 S_{t,T}^x}{|D_v S_{t,T}^x|^2}\right) \\
&\quad \left.\left. + 2b_T'(S_{t,T}^x)^2 + 2b_T''(S_{t,T}^x)b_T(S_{t,T}^x)\right)\right].
\end{aligned}$$

Summing (4.5) with the above relation and using (4.3) we obtain

$$\begin{aligned}
\Lambda_{t,T}(u,v,w) &= -a_T'(S_{t,T}^x) - r_T + a_T(S_{t,T}^x)\left(\frac{I_1(u)}{D_u S_{t,T}^x} + \frac{D_u^2 S_{t,T}^x}{|D_u S_{t,T}^x|^2}\right) \\
&+ \frac{1}{2}\left(\Gamma_{t,T}(v)\left(\frac{I_1(w)}{D_w S_{t,T}^x} + \frac{D_w^2 S_{t,T}^x}{|D_w S_{t,T}^x|^2}\right) - \frac{2b_T'(S_{t,T}^x)b_T(S_{t,T}^x)}{D_v S_{t,T}^x}\left(I_1(v) + \frac{D_v^2 S_{t,T}^x}{D_v S_{t,T}^x}\right)\right. \\
&+ \left.\frac{b_T^2(S_{t,T}^x)}{D_w S_{t,T}^x D_v S_{t,T}^x}\left(I_1(v)\frac{D_w D_v S_{t,T}^x}{D_v S_{t,T}^x} - \langle v, w\rangle - \frac{D_w D_v^2 S_{t,T}^x}{D_v S_{t,T}^x} + \frac{2D_w D_v S_{t,T}^x D_v^2 S_{t,T}^x}{|D_v S_{t,T}^x|^2}\right)\right) \\
&+ b_T'(S_{t,T}^x)^2 + b_T''(S_{t,T}^x)b_T(S_{t,T}^x).
\end{aligned}$$

We conclude the proof by approximation of ϕ by $\mathcal{C}_b^2$ functions. $\square$

In the time homogeneous case, denoting $S_{0,\tau}^x$ by S_τ^x and $\Lambda_{0,\tau}(u,v,w)$ by $\Lambda_\tau(u,v,w)$ we have

$$C(x,t,T) = e^{-(T-t)r}\mathbb{E}\left[\phi(S_T)\Big|S_t = x\right] = e^{-\tau r}\mathbb{E}\left[\phi(S_\tau^x)\right],$$

with $\tau = T - t$, and

$$\begin{aligned}&\mathrm{Theta}_T = -\mathrm{Theta}_t \\ &= \ e^{-\tau r}\mathbb{E}\left[\left(\Lambda_\tau(u,v,w)\phi(S_\tau^x) + \lambda\int_{-\infty}^{+\infty}(\phi(S_\tau^x + c(S_\tau^x)y) - \phi(S_\tau^x))\nu(dy)\right)\right],\end{aligned}$$

where $u,v,w \in L^2([0,\tau])$ are such that $D_uS_\tau^x$, $D_vS_\tau^x$, $D_wS_\tau^x$ are a.s. non-zero and $\Lambda_\tau(u,v,w) \in L^2(\Omega)$. For constant r, σ and η, i.e. in the linear case, we have

$$\begin{cases} a(y) = (r - \lambda\eta)y, \\ b(y) = \sigma y, \\ c(y) = \eta y, \end{cases}$$

and (2.7) yields

$$D_uS_\tau^x = \sigma\int_0^\tau u_s ds S_\tau^x,$$

hence $D_v^2S_\tau^x/|D_vS_\tau^x|^2 = 1/S_\tau$ and we get

$$\Lambda_\tau(u,v,w) = -r + \frac{\hat{r}}{\sigma}\frac{I_1(u)}{\int_0^\tau u_s ds} - \frac{\sigma}{2}\frac{I_1(w)}{\int_0^\tau w_s ds} + \frac{I_2(v\circ w)}{2\int_0^\tau v_s ds\int_0^\tau w_s ds},$$

where $\hat{r} = r - \lambda\eta$. Theta is then given by

$$\mathrm{Theta} = e^{-r\tau}\mathbb{E}\left[\Lambda_\tau(u,v,w)\phi(S_\tau^x) + \lambda\int_{-\infty}^{+\infty}(\phi(S_\tau^x(1+\eta y)) - \phi(S_\tau^x))\nu(dy)\right],$$

i.e. in the model of (2.5) we have

$$\mathrm{Theta} = e^{-r\tau}\mathbb{E}\left[\Lambda_\tau(u,v,w)\phi(S_\tau^x) + \sum_{k=1}^{d}\lambda_k(\phi(S_\tau^x(1+\eta a_k)) - \phi(S_\tau^x))\right],$$

and if $(X_t)_{t\in\mathbb{R}_+}$ is a standard Poisson process with intensity λ and jump size a, we get

$$\mathrm{Theta} = e^{-r\tau}\mathbb{E}\left[\Lambda_\tau(u,v,w)\phi(S_\tau^x) + \lambda(\phi((1+a\eta)S_\tau^x) - \phi(S_\tau^x))\right].$$

If $(X_t)_{t\in\mathbb{R}_+}$ has infinitely many jumps on bounded time intervals, i.e. if $\mu(\mathbb{R}) = \infty$, then $(S_t)_{t\in\mathbb{R}_+}$ is given by

$$\begin{cases} dS_t = a_t(S_t)dt + b_t(S_t)dW_t + c_t(S_{t^-})d\tilde{X}_t, \\ \\ S_0 = x, \end{cases}$$

where $(\tilde{X}_t)_{t\in\mathbb{R}_+}$ is the compensated pure jump martingale associated to $(X_t)_{t\in\mathbb{R}_+}$. In this case Itô's formula reads

$$\begin{aligned} \phi(S_t) &= \phi(S_s) + \int_s^t \phi'(S_u)a_u(S_u)du + \int_s^t \phi'(S_u)b_u(S_u)dW_u + \int_s^t \phi'(S_u)b_u(S_u)d\tilde{X}_u \\ &+\frac{1}{2}\int_s^t \phi''(S_u)b^2(S_u)du + \sum_{s<u\le t}\left(\phi(S_{u^-} + c_u(S_{u^-})\Delta X_u) - \phi(S_{u^-}) - c_u(S_{u^-})\Delta X_u\phi'(S_{u^-})\right), \end{aligned}$$

$0 \le s \le t$, to yield

$$\begin{aligned} \mathbb{E}[\phi(S_t)] &= \mathbb{E}[\phi(S_s)] + \mathbb{E}\left[\int_s^t \phi'(S_u)a_u(S_u)du\right] + \frac{1}{2}\mathbb{E}\left[\int_s^t \phi''(S_u)b_u^2(S_u)du\right] \\ &\quad +\lambda\mathbb{E}\left[\int_s^t\int_{-\infty}^{+\infty}(\phi(S_u + zc_u(S_u)) - \phi(S_u) - zc_u(S_u)\phi'(S_u))\nu(dz)du\right]. \end{aligned}$$

However, in this case the last component in $\phi'(S_u)$ can not be isolated and dealt with by integration by parts.

5 Optimization of convergence

By the results of the preceding section, the value of Theta in the geometric model (2.6) is given by

$$\text{Theta} = e^{-r\tau}\mathbb{E}\left[\phi(S_\tau^x)\Lambda(u,v,w) + \lambda\int_{-\infty}^{+\infty}(\phi(S_\tau^x + \eta S_\tau^x y) - \phi(S_\tau^x))\nu(dy)\right]. \quad (5.1)$$

For any $u \in L^2([0,\tau])$ such that $\int_0^\tau u_s ds \neq 0$, letting

$$\tilde{u}_t = \frac{u_t}{\int_0^\tau u_s ds}, \qquad t \in [0,\tau],$$

the weight $\Lambda_\tau(u,v,w)$ is expressed as

$$\Lambda_\tau(u,v,w) = -r + \frac{\hat{r}}{\sigma} I_1(\tilde{u}) - \frac{\sigma}{2} I_1(\tilde{w}) + \frac{1}{2} I_2(\tilde{v} \circ \tilde{w}).$$

Our goal is now to find functions u,v,w which minimize $\mathrm{Var}[\Lambda_\tau(u,v,w)]$ in this setting.

Proposition 5.1 *The infimum on* $\mathrm{Var}[\Lambda_\tau(u,v,w)]$ *is attained for any non-zero constant functions* u,v,w *of the form* $u_s = c_1$, $v_s = c_2$, $w_s = c_3$, $s \in [0,\tau]$, *and is given by*

$$\inf_{u,v,w} \mathrm{Var}[\Lambda_\tau(u,v,w)] = \mathrm{Var}[\Lambda_{\mathrm{opt}}] = \frac{1}{2\tau^2} + \frac{1}{\sigma^2 \tau} \left| \hat{r} - \frac{\sigma^2}{2} \right|^2,$$

where $\hat{r} = r - \lambda\eta$, *with*

$$\Lambda_{\mathrm{opt}} = -r + \frac{W_\tau}{\sigma\tau} \left(\hat{r} - \frac{\sigma^2}{2} \right) + \frac{1}{2\tau} \left(\frac{W_\tau^2}{\tau} - 1 \right). \tag{5.2}$$

Proof. Recall that the Cauchy-Schwarz inequality yields

$$\|\tilde{u}\|^2 \geq \frac{1}{\tau}, \tag{5.3}$$

with equality if and only if $\tilde{u}_t = 1/\tau$, $t \in [0,\tau]$.

Let

$$\begin{aligned} F(u,v,w) &= \mathrm{Var}[\Lambda_\tau(u,v,w)] \\ &= \frac{\hat{r}^2}{\sigma^2} \|\tilde{u}\|^2 - \hat{r} \langle \tilde{u}, \tilde{w} \rangle + \frac{\sigma^2}{4} \|\tilde{w}\|^2 + \frac{1}{4} \|\tilde{v}\|^2 \|\tilde{w}\|^2 + \frac{1}{4} \langle \tilde{v}, \tilde{w} \rangle^2 \\ &= \frac{1}{\sigma^2} \left\| \hat{r}\tilde{u} - \frac{\sigma^2}{2} \tilde{w} \right\|^2 + \frac{1}{4} \|\tilde{v}\|^2 \|\tilde{w}\|^2 + \frac{1}{4} \langle \tilde{v}, \tilde{w} \rangle^2. \end{aligned}$$

The optimal value of (u,v,w) solves

$$\begin{cases} \dfrac{d}{d\varepsilon} F(u + \varepsilon h, v, w)_{|\varepsilon=0} = 0 \\[2ex] \dfrac{d}{d\varepsilon} F(u, v + \varepsilon h, w)_{|\varepsilon=0} = 0 \\[2ex] \dfrac{d}{d\varepsilon} F(u, v, w + \varepsilon h)_{|\varepsilon=0} = 0, \end{cases} \tag{5.4}$$

for all $h \in L^2([0,\tau])$, i.e.

$$\frac{2\hat{r}^2}{\sigma^2}\left(\langle h,\tilde{u}\rangle - \|\tilde{u}\|^2\int_0^\tau h_s ds\right) - \hat{r}\left(\langle h,\tilde{w}\rangle - \langle \tilde{u},\tilde{w}\rangle\int_0^\tau h_s ds\right) = 0,$$

$$\frac{1}{2}\|\tilde{w}\|^2\left(\langle h,\tilde{v}\rangle - \|\tilde{v}\|^2\int_0^\tau h_s ds\right) + \frac{1}{2}\left(\langle \tilde{v},\tilde{w}\rangle\langle h,\tilde{w}\rangle - \langle \tilde{v},\tilde{w}\rangle^2\int_0^\tau h_s ds\right) = 0,$$

and

$$\frac{\sigma^2}{2}\left(\langle h,\tilde{w}\rangle - \|\tilde{w}\|^2\int_0^\tau h_s ds\right) + \frac{1}{2}\|\tilde{v}\|^2\left(\langle h,\tilde{w}\rangle - \|\tilde{w}\|^2\int_0^\tau h_s ds\right)$$
$$+\frac{1}{2}\left(\langle \tilde{v},\tilde{w}\rangle\langle h,\tilde{v}\rangle - \langle \tilde{v},\tilde{w}\rangle^2\int_0^\tau h_s ds\right) - \hat{r}\left(\langle h,\tilde{u}\rangle - \langle \tilde{u},\tilde{w}\rangle\int_0^\tau h_s ds\right) = 0.$$

Clearly, for all $c_1, c_2, c_3 \neq 0$ the constant functions $u_s = c_1$, $v_s = c_2$, $w_s = c_3$, $s \in [0,\tau]$, are solutions of this problem. Let us show that this solution is unique. For all $h \in L^2([0,\tau])$ such that $\int_0^\tau h_s ds = 0$, equation (5.4) yields

$$\begin{cases} \dfrac{2\hat{r}^2}{\sigma^2}\langle h,\tilde{u}\rangle - \hat{r}\langle h,\tilde{w}\rangle = 0 \\ \\ \|\tilde{w}\|^2\langle h,\tilde{v}\rangle + \langle \tilde{v},\tilde{w}\rangle\langle h,\tilde{w}\rangle = 0 \\ \\ \sigma^2\langle h,\tilde{w}\rangle + \|\tilde{v}\|^2\langle h,\tilde{w}\rangle + \langle \tilde{v},\tilde{w}\rangle\langle h,\tilde{v}\rangle - 2\hat{r}\langle h,\tilde{u}\rangle = 0. \end{cases}$$

If a solution $(\tilde{u},\tilde{v},\tilde{w})$ different from $(1/\tau,1/\tau,1/\tau)$ exists, then one can find $h \in L^2([0,\tau])$ such that $\int_0^\tau h_s ds = 0$ and $(\langle h,\tilde{u}\rangle,\langle h,\tilde{v}\rangle,\langle h,\tilde{w}\rangle) \neq (0,0,0)$, hence the determinant

$$\|\tilde{v}\|^2\|\tilde{w}\|^2 - |\langle \tilde{v},\tilde{w}\rangle|^2 = 0 \tag{5.5}$$

of the above linear system vanishes. From (5.3) and (5.5) we get

$$\begin{aligned} F(u,v,w) &= \frac{1}{\sigma^2}\left\|\hat{r}\tilde{u} - \frac{\sigma^2}{2}\tilde{w}\right\|^2 + \frac{1}{4}\|\tilde{v}\|^2\|\tilde{w}\|^2 + \frac{1}{4}|\langle \tilde{v},\tilde{w}\rangle|^2 \\ &= \frac{1}{\sigma^2}\left\|\hat{r}\tilde{u} - \frac{\sigma^2}{2}\tilde{w}\right\|^2 + \frac{1}{2}\|\tilde{v}\|^2\|\tilde{w}\|^2 \\ &\geq \frac{1}{\tau\sigma^2}\left|\int_0^\tau \left(\hat{r}\tilde{u}_s - \frac{\sigma^2}{2}\tilde{w}_s\right) ds\right|^2 + \frac{1}{2\tau^2} \end{aligned}$$

$$= \frac{1}{\tau\sigma^2}\left|\hat{r}-\frac{\sigma^2}{2}\right|^2+\frac{1}{2\tau^2},$$

which is greater than the optimal value found when $\tilde{u}$, $\tilde{v}$, $\tilde{w}$ are constant functions. Moreover, equality occurs only when $\|\tilde{v}\|^2 = 1/\tau$, $\|\tilde{w}\|^2 = 1/\tau$, and

$$\left\|\hat{r}\tilde{u}-\frac{\sigma^2}{2}\tilde{w}\right\|^2 = \frac{1}{\tau}\left|\hat{r}-\frac{\sigma^2}{2}\right|^2,$$

i.e. when $\hat{r}\tilde{u}-\frac{\sigma^2}{2}\tilde{w}$, $\tilde{v}$, $\tilde{w}$ are constant, which implies that $\tilde{u}$ is also constant, except when $\hat{r}=0$, in which case no constraint is imposed on u.

We now need to prove that this solution corresponds to the global minimum of F. Since $F(u,v,w)\geq 0$, the infimum exists and we denote it by l. By continuity of F on $L^2([0,\tau])^3$ there exist a sequence $(u_n,v_n,w_n)_{n\in\mathbb{N}}$ such that

$$l = \lim_{n\to\infty}\mathbb{E}[\Lambda_\tau(u_n,v_n,w_n)^2].$$

We can assume that (u_n,v_n,w_n) is bounded: if not, replace it by the bounded sequence

$$\left(\frac{u_n}{\|u_n\|},\frac{v_n}{\|v_n\|},\frac{w_n}{\|w_n\|}\right)_{n\in\mathbb{N}},$$

on which F takes the same values as on $(u_n,v_n,w_n)_{n\in\mathbb{N}}$. Under this hypothesis, there exists a sub-sequence $(u_{n_k},v_{n_k},w_{n_k})_{k\in\mathbb{N}}$ converging weakly to (u,v,w) in $L^2([0,\tau])^3$. We have

$$\begin{aligned}
&\mathbb{E}[\Lambda_\tau(u,v,w)\Lambda_\tau(u_{n_k},v_{n_k},w_{n_k})]\\
&\qquad = \hat{r}^2+\frac{\hat{r}^2}{\sigma^2}\langle\tilde{u},\tilde{u}_{n_k}\rangle-\frac{\hat{r}}{2}\langle\tilde{u},\tilde{w}_{n_k}\rangle-\frac{\hat{r}}{2}\langle\tilde{u}_{n_k},\tilde{w}\rangle+\frac{\sigma^2}{4}\langle\tilde{w},\tilde{w}_{n_k}\rangle\\
&\qquad\quad +\frac{1}{4}\langle\tilde{u},\tilde{u}_{n_k}\rangle\langle\tilde{w},\tilde{w}_{n_k}\rangle+\frac{1}{4}\langle\tilde{u},\tilde{w}_{n_k}\rangle\langle\tilde{w},\tilde{u}_{n_k}\rangle,
\end{aligned}$$

and by weak convergence of $(u_{n_k},v_{n_k},w_{n_k})_{k\in\mathbb{N}}$ to (u,v,w) we get

$$\lim_{n\to\infty}\mathbb{E}[\Lambda_\tau(u,v,w)\Lambda_\tau(u_{n_k},v_{n_k},w_{n_k})] = \mathbb{E}[|\Lambda_\tau(u,v,w)|^2].$$

Moreover,

$$0 \geq l-\mathbb{E}[|\Lambda_\tau(u,v,w)|^2]$$

$$
\begin{aligned}
&= \lim_{n\to\infty} \mathbb{E}[\Lambda_\tau(u_{n_k}, v_{n_k}, w_{n_k})^2] - \mathbb{E}[|\Lambda_\tau(u,v,w)|^2] \\
&\geq \lim_{n\to\infty} \mathbb{E}[|\Lambda_\tau(u,v,w) - \Lambda_\tau(u_{n_k}, v_{n_k}, w_{n_k})|^2] \\
&\quad +2\mathbb{E}[\Lambda_\tau(u,v,w)\Lambda_\tau(u_{n_k}, v_{n_k}, w_{n_k})] - 2\mathbb{E}[|\Lambda_\tau(u,v,w)|^2] \\
&\geq \lim_{n\to\infty} \mathbb{E}[(\Lambda_\tau(u,v,w) - \Lambda_\tau(u_{n_k}, v_{n_k}, w_{n_k}))^2] \\
&\quad +2\lim_{n\to\infty} \mathbb{E}[\Lambda_\tau(u,v,w)\Lambda_\tau(u_{n_k}, v_{n_k}, w_{n_k})] \\
&\quad -2\mathbb{E}[|\Lambda_\tau(u,v,w)|^2] \\
&\geq \lim_{n\to\infty} \mathbb{E}[(\Lambda_\tau(u,v,w) - \Lambda_\tau(u_{n_k}, v_{n_k}, w_{n_k}))^2] \\
&\geq 0,
\end{aligned}
$$

hence $\lim_{n\to\infty} \Lambda_\tau(u_{n_k}, v_{n_k}, w_{n_k}) = \Lambda_\tau(u,v,w)$ in $L^2(\Omega)$ and

$$l = \mathbb{E}[|\Lambda_\tau(u,v,w)|^2].$$

Thus the global minimum is attained for $\tilde{u} = \tilde{v} = \tilde{w} = 1/\tau$. □

Note that $\inf_{u,v,w\in L^2([0,\tau])} \mathrm{Var}[\Lambda_\tau(u,v,w)]$ is minimal in terms of σ and r when $(S_t^x)_{t\in\mathbb{R}_+}$ is an exponential Brownian motion, i.e. $\hat{r} = \sigma^2/2$. In this case we have

$$\inf_{u,v,w\in L^2([0,\tau])} \mathrm{Var}[\Lambda_\tau(u,v,w)] = \frac{1}{2\tau^2}.$$

6 Asian options

Consider the price process $(S_t)_{t\in\mathbb{R}_+}$ given by

$$\begin{cases} dS_t^x = a(S_t^x)dt + b(S_t^x)dW_t + \eta S_{t^-}^x dX_t, \\ \\ S_0^x = x, \end{cases}$$

i.e. we let $c_t(z) = \eta z$. Recall that, cf. [11], the price

$$C(x,y,t,T) = e^{-(T-t)r}\mathbb{E}\left[\left(\frac{1}{T}\int_0^T S_u^x du - K\right)^+ \Big| \mathcal{F}_t\right]$$

at time t of an Asian option with price process $(S_t^x)_{t\in\mathbb{R}_+}$ can be expressed as

$$C(x,y,t,T) = xe^{-(T-t)r}\mathbb{E}\left[\left(y + \frac{1}{T}\int_t^T S_{s-t}^1 ds\right)^+\right] \tag{6.1}$$

and

$$x = S_t \qquad \text{and} \qquad y = \frac{1}{x}\left(\frac{1}{T}\int_0^t S_u du - K\right).$$

As in the setting of European options, we may consider

$$\text{Theta}_t = \frac{\partial C}{\partial t}(x,y,t,T) \qquad \text{and} \qquad \text{Theta}_T = \frac{\partial C}{\partial T}(x,y,t,T),$$

which however differ from each other in general, including when $(S_t)_{t\in\mathbb{R}_+}$ is time homogeneous. Here, Theta_t can be computed from the Black-Scholes PDE for Asian options in a jump-diffusion model, cf. Proposition 9 of [1], as

$$\begin{aligned}\text{Theta}_t &= rC(x,y,t,T) + \left(\frac{1}{T} - a(y)\right)\frac{\partial C}{\partial y}(x,y,t,T) - \frac{1}{2}b^2(y)\frac{\partial^2 C}{\partial y^2}(x,y,t,T) \\ &\quad -(1+\eta)\lambda\int_{-\infty}^{+\infty}\left(C\left(x,\frac{yz}{1+\eta},t,T\right) - C(x,y,t,T)\right)\nu(dz).\end{aligned}$$

In the setting of Asian options, our method applies both to Theta_t and Theta_T and does not require the first and second derivatives $\dfrac{\partial C}{\partial y}$ and $\dfrac{\partial^2 C}{\partial y^2}$. It also does not use Itô calculus.

Proposition 6.1 *Given $u \in L^2([0,\tau])$ such that $\int_0^\tau D_u S_s^x ds \neq 0$, a.s., $s \in [0,\tau]$, let*

$$\Lambda(u) = -r - \frac{D_u S_\tau^x}{\int_0^\tau D_u S_s^x ds} + \frac{S_\tau^x - \int_0^\tau S_s^x ds/T}{\int_0^\tau D_u S_s^x ds}\left(I_1(u) + \frac{\int_0^\tau D_u^2 S_s^x ds}{\int_0^\tau D_u S_s^x ds}\right),$$

and assume that $\Lambda(u) \in L^2(\Omega)$. We have

$$\text{Theta}_T = xe^{-(T-t)r}\mathbb{E}\left[\Lambda(u)\left(y + \frac{1}{T}\int_0^\tau S_s^1 ds\right)^+\right].$$

Proof. Replacing $x \mapsto (x-K)^+$ in (6.1) by ϕ a $\mathcal{C}_b^1$ function we define the time derivative

$$\begin{aligned}\text{Theta}_T^\phi &= \frac{\partial C}{\partial T}(x,y,t,T) \\ &= xe^{-(T-t)r}\mathbb{E}\left[-r\phi\left(y + \frac{1}{T}\int_t^T S_{s-t}^1 ds\right)\right. \\ &\quad \left. + \left(\frac{S_{T-t}^1}{T} - \frac{1}{T^2}\int_t^T S_{s-t}^1 ds\right)\phi'\left(y + \frac{1}{T}\int_t^T S_{s-t}^1 ds\right)\right].\end{aligned}$$

Using (3.5) we have :

$$\begin{aligned}
\mathrm{Theta}_T^\phi &= -rxe^{-(T-t)r}\mathbb{E}\left[\phi\left(y+\frac{1}{T}\int_0^\tau S_s^1ds\right)\right] \\
&\quad +xe^{-(T-t)r}\mathbb{E}\left[\left(S_\tau^1-\frac{1}{T}\int_0^\tau S_s^1ds\right)\frac{D_u\phi\left(y+\int_0^\tau S_s^1ds/T\right)}{D_u\int_0^\tau S_s^1ds}\right] \\
&= -rxe^{-(T-t)r}\mathbb{E}\left[\phi\left(y+\frac{1}{T}\int_0^\tau S_s^1ds\right)\right] \\
&\quad +xe^{-(T-t)r}\mathbb{E}\left[\left\langle D\phi\left(y+\frac{1}{T}\int_0^\tau S_s^1ds\right), \frac{S_\tau^1-\int_0^\tau S_s^1ds/T}{\int_0^\tau D_uS_s^1ds}u\right\rangle\right] \\
&= -rxe^{-(T-t)r}\mathbb{E}\left[\phi\left(y+\frac{1}{T}\int_0^\tau S_s^1ds\right)\right] \\
&\quad +xe^{-(T-t)r}\mathbb{E}\left[\phi\left(y+\frac{1}{T}\int_0^\tau S_s^1ds\right)\delta\left(\frac{S_\tau^1-\int_0^\tau S_s^1ds/T}{\int_0^\tau D_uS_s^1ds}u\right)\right] \\
&= xe^{-(T-t)r}\mathbb{E}\left[\phi\left(y+\frac{1}{T}\int_0^\tau S_s^1ds\right)\left(-r-\frac{D_uS_\tau^1}{\int_0^\tau D_uS_s^1ds}\right.\right. \\
&\quad \left.\left.+\frac{S_\tau^1-\int_0^\tau S_s^1ds/T}{\int_0^\tau D_uS_s^1ds}\left(I_1(u)+\frac{\int_0^\tau D_u^2S_s^1ds}{\int_0^\tau D_uS_s^1ds}\right)\right)\right] \\
&= xe^{-(T-t)r}\mathbb{E}\left[\Lambda(u)\phi\left(y+\frac{1}{T}\int_0^\tau S_s^1ds\right)\right].
\end{aligned}$$

It remains to approximate $x\mapsto(x-K)^+$ by a sequence $(\phi_n)_{n\in\mathbb{N}}$ of $\mathcal{C}_b^1$ functions. □

Theta_t can be computed in a similar way from

$$C(x,y,t,T)=xe^{-(T-t)r}\mathbb{E}\left[\left(y+\frac{1}{T}\int_0^{T-t}S_s^1ds\right)^+\right],$$

and

$$\mathrm{Theta}_t^\phi=xe^{-(T-t)r}\mathbb{E}\left[r\phi\left(y+\frac{1}{T}\int_0^{T-t}S_{t-s}^1ds\right)+\frac{S_{T-t}^1}{T}\phi'\left(y+\frac{1}{T}\int_0^{T-t}S_s^1ds\right)\right],$$

which yields after integration by parts:

$$\mathrm{Theta}_t=xe^{-(T-t)r}\mathbb{E}\left[\tilde{\Lambda}(u)\left(y+\frac{1}{T}\int_0^{T-t}S_{t-s}^1ds\right)^+\right],$$

with

$$\tilde{\Lambda}(u) = r - \frac{D_u S_\tau^x}{\int_0^\tau D_u S_s^x ds} + \frac{S_\tau^x}{\int_0^\tau D_u S_s^x ds}\left(I_1(u) + \frac{\int_0^\tau D_u^2 S_s^x ds}{\int_0^\tau D_u S_s^x ds}\right).$$

In the geometric model (2.6) we have

$$\Lambda(u) = -r - \frac{\int_0^\tau u_s ds S_\tau^x}{\int_0^\tau \int_0^s u_p dp S_s^x ds} + \frac{S_\tau^x - \int_0^\tau S_s^x ds/T}{\sigma \int_0^\tau \int_0^s u_p dp S_s^x ds}\left(I_1(u) + \sigma\frac{\int_0^\tau \left(\int_0^s u_p dp\right)^2 S_s^x ds}{\int_0^\tau \int_0^s u_p dp S_s^x ds}\right)$$

and

$$\tilde{\Lambda}(u) = r - \frac{\int_0^\tau u_s ds S_\tau^x}{\int_0^\tau \int_0^s u_p dp S_s^x ds} + \frac{S_\tau^x}{\sigma \int_0^\tau \int_0^s u_p dp S_s^x ds}\left(I_1(u) + \sigma\frac{\int_0^\tau \left(\int_0^s u_p dp\right)^2 S_s^x ds}{\int_0^\tau \int_0^s u_p dp S_s^x ds}\right).$$

7 Numerical simulations

In this section we consider the geometric Brownian model

$$\begin{cases} dS_s = rS_s ds + \sigma S_s dW_s \\ S_t = x, \end{cases}$$

and apply the Malliavin formula (5.1) with $\tilde{u}_s = \tilde{v}_s = \tilde{w}_s = 1/\tau$, $s \in [0,\tau]$, to compute

$$\mathrm{Theta}_T = -\mathrm{Theta}_t$$

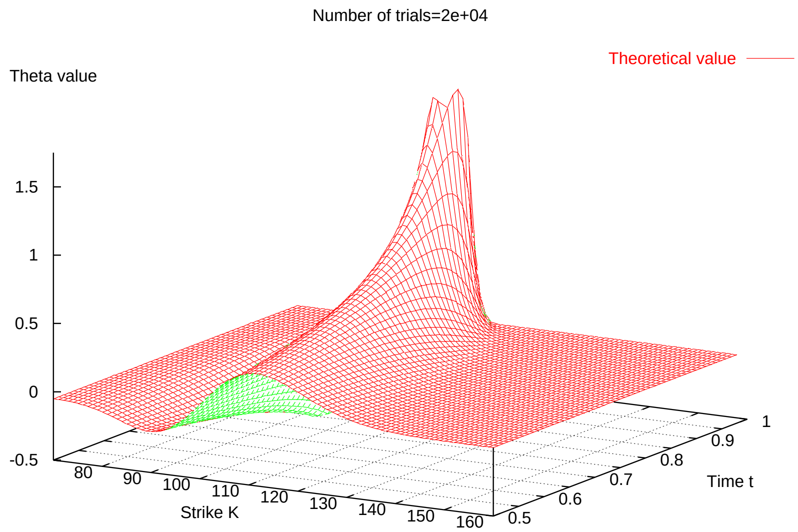
for European type options with non-smooth payoff functions. Formula (4.1) is used for finite differences approximations.

7.1 Digital options

In case $\phi = \mathbf{1}_{[K,+\infty)}$, the value of Theta can be computed analytically as

$$\mathrm{Theta} = e^{-r\tau}\left(e^{-\frac{a^2}{2\tau}}\left(\frac{r}{\sigma} - \frac{\sigma}{2} + \frac{a}{2\tau\sqrt{2\pi\tau}}\right) - r\int_{\frac{a}{\sqrt{\tau}}}^\infty \frac{e^{-\frac{y^2}{2}}}{\sqrt{2\pi}} dy\right),$$

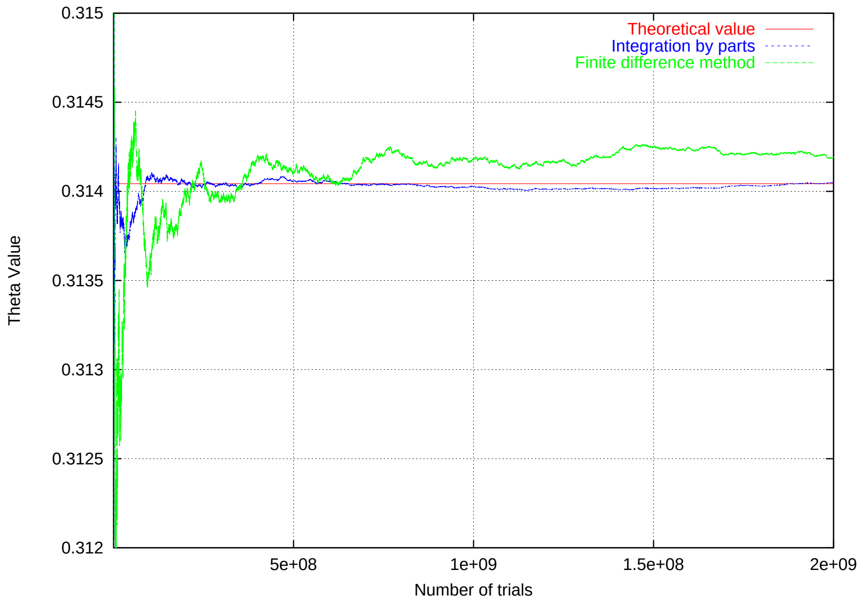
with $a = (\ln(K/x) - r\tau)/\sigma$, cf. e.g. [8], which yields the following graph:



T=1, x=100, σ=0.15, r=0.05, $\varepsilon = 0.001$

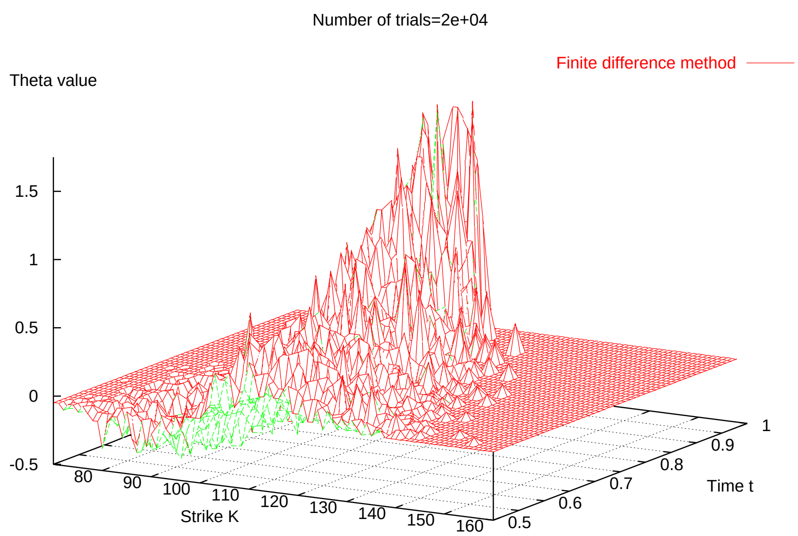
Figure 1 : Exact value of Theta as a function of t and K

The next graphs allow us to compare the Monte Carlo simulations obtained by finite differences and integration by parts, first in terms of number of samples and then as functions of t and K.



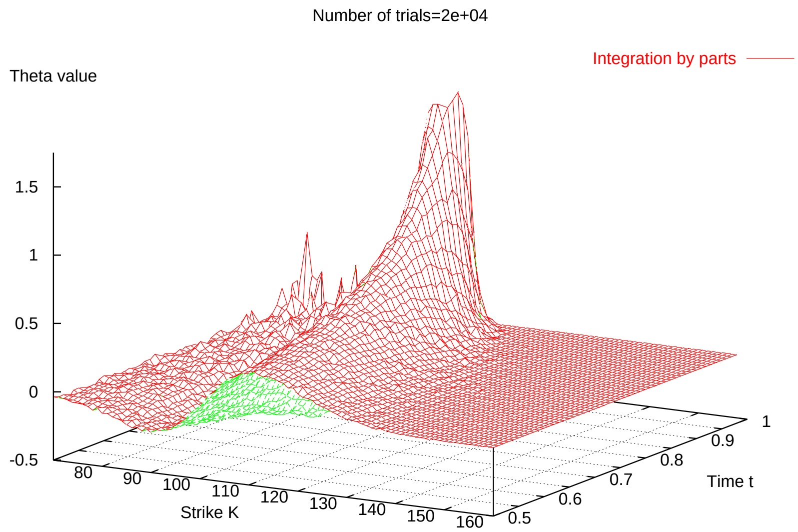
T=1, t=0.5, x=100, σ=0.2, r=0.1, K=110, ε=0.01

Figure 2 : Estimation of Theta vs number of trials



T=1, x=100, σ=0.15, r=0.05, ε=0.001

Figure 3 : Theta as a function of t and K by finite differences



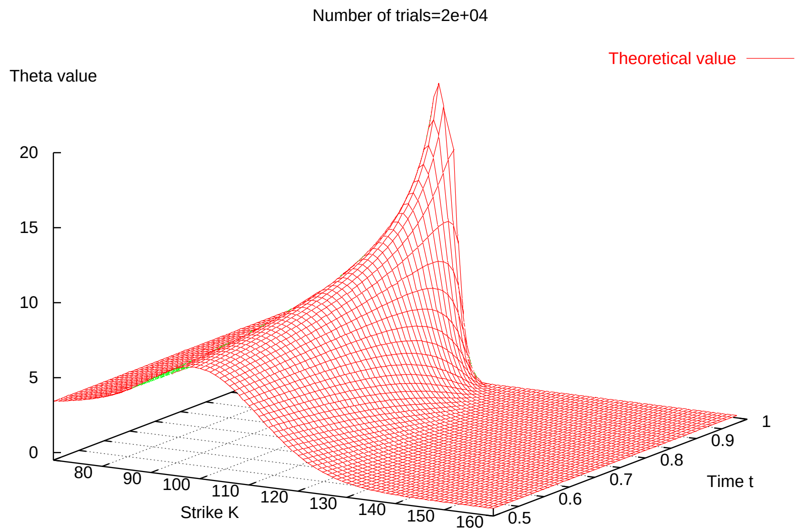
$T{=}1$, $x{=}100$, $\sigma{=}0.15$, $r{=}0.05$, $\varepsilon{=}0.001$
Figure 4 : Theta as a function of t and K by integration by parts

7.2 European options

In case $\phi(x) = (x-K)^+$, Theta can be computed as

$$\text{Theta} = \frac{x\sigma}{2\sqrt{2\pi\tau}} e^{-c^2/2} + Kre^{-r\tau} N(c - \sigma\sqrt{\tau}),$$

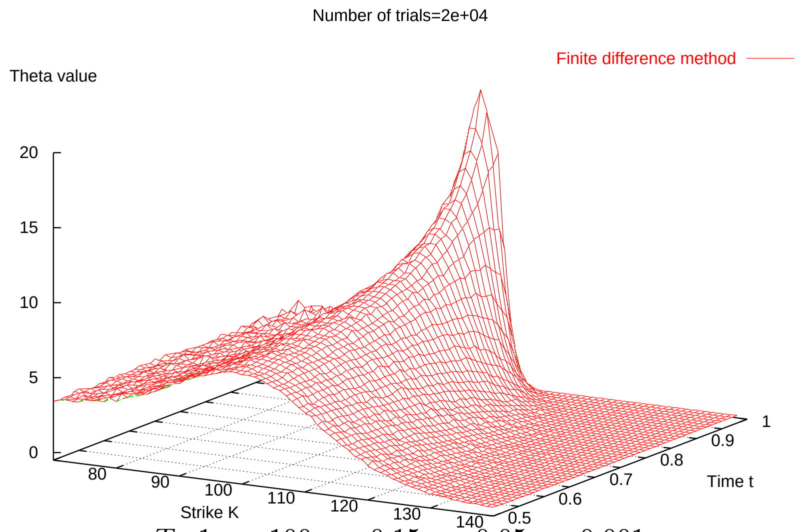
where $c = \left(\sigma - \frac{a}{\tau}\right)\sqrt{\tau}$ and $N(x) = \int_{-\infty}^{x} e^{-\frac{y^2}{2}} \frac{dy}{\sqrt{2\pi}}$, cf. e.g. [8], which yields the following graph.



T=1, x=100, σ=0.15, r=0.05, ε=0.001

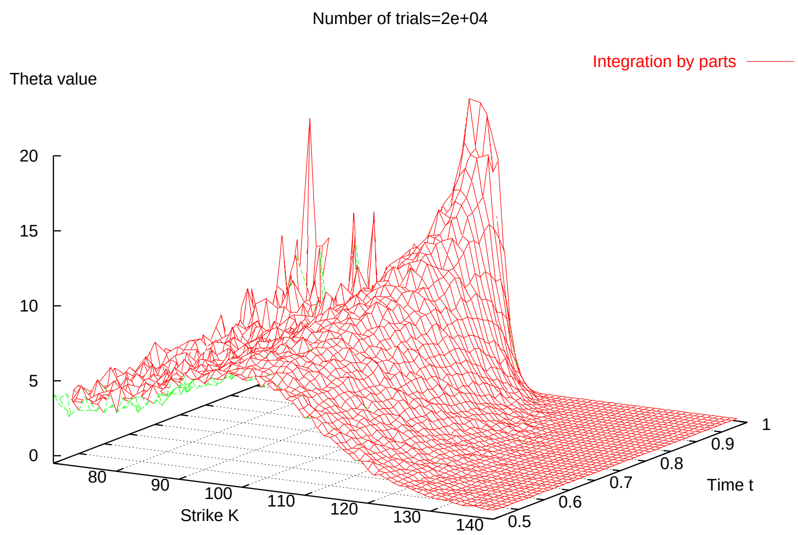
Figure 5 : Exact value of Theta as a function of t and K

As in [4], the following graphs show that the finite difference approximation performs better for European calls with $\phi(x) = (x - K)^+$, but as seen in the previous section, the Malliavin calculus approach performs better for digital options.



T=1, x=100, σ=0.15, r=0.05, ε=0.001

Figure 6 : Theta as a function of t and K by finite differences



T=1, x=100, σ=0.15, r=0.05, ε=0.001

Figure 7 : Theta as a function of t and K by integration by parts

The localization procedure applied in the next section will allow us to improve the result of the Malliavin method.

8 Localization

Payoff functions of the form

$$\phi(y) = \mathbf{1}_{[K,+\infty)}(y) \qquad \text{and} \qquad \phi(y) = (y-K)^+$$

have a singularity at $y = K$. The idea of localization is to decompose the payoff function ϕ as

$$\phi = g_\varepsilon + h_\varepsilon$$

in such a way that h_ε is twice differentiable and g_ε contains the singularity of ϕ, see e.g. [9] for digital options and [2] for European options. Applying the Malliavin approach to g_ε and using (4.3) we get

$$\begin{aligned}
\text{Theta} &= e^{-\tau r}\mathbb{E}\left[\left(\Lambda(u,v,w)g_\varepsilon(S_\tau^x) + \lambda\int_{-\infty}^{+\infty}(\phi(S_\tau^x + c(S_\tau^x)y) - \phi(S_\tau^x))\nu(dy)\right)\right] \\
&\quad -re^{-\tau r}\mathbb{E}\left[h_\varepsilon(S_\tau^x)\right] + e^{-\tau r}\mathbb{E}\left[h_\varepsilon'(S_\tau^x)a(S_\tau^x)\right] \\
&\quad +\frac{1}{2}e^{-\tau r}\mathbb{E}\left[h_\varepsilon''(S_\tau^x)b^2(S_\tau^x)\right].
\end{aligned}$$

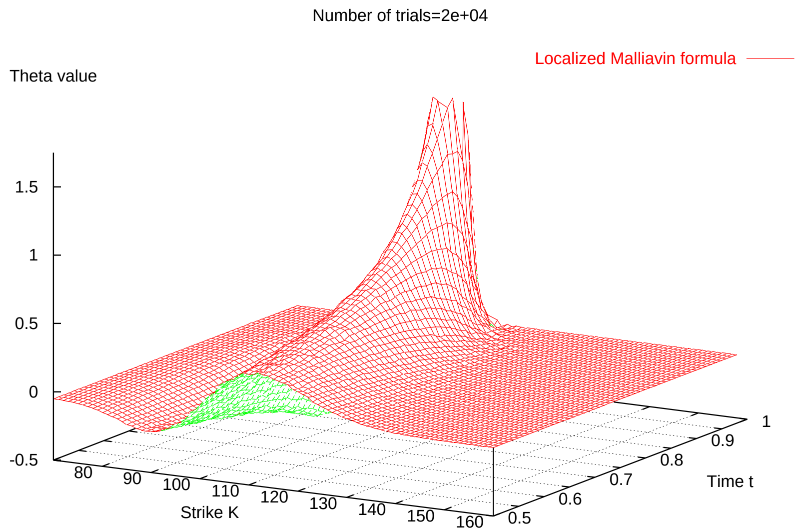
We applied the integration by parts method to get rid of the first and second derivatives on g_ε. In the geometric model with the optimal weight $\Lambda(u,v,w)$ this gives:

$$\begin{aligned}
\text{Theta} &= -re^{-r\tau}\mathbb{E}\left[\phi(S_\tau^x)\right] \\
&\quad +r\frac{e^{-r\tau}}{\sigma\tau}\mathbb{E}\left[g_\varepsilon(S_\tau^x)W_\tau\right] + \frac{e^{-r\tau}}{2\tau}\mathbb{E}\left[g_\varepsilon(S_\tau^x)\left(\frac{W_\tau^2}{\tau} - \sigma W_\tau - 1\right)\right] \\
&\quad +re^{-r\tau}\mathbb{E}\left[S_\tau^x h_\varepsilon'(S_\tau^x)\right] + \frac{\sigma^2}{2}e^{-r\tau}\mathbb{E}\left[S_\tau^{x2} h_\varepsilon''(S_\tau^x)\right].
\end{aligned}$$

For digital options we take

$$\begin{aligned}
h_\varepsilon(y) &= \frac{1}{2}\left(1+\frac{y-K}{\varepsilon}\right)^2\mathbf{1}_{]-\varepsilon,0]}(y-K) + \left(1-\frac{1}{2}\left(1-\frac{y-k}{\varepsilon}\right)^2\right)\mathbf{1}_{]0,\varepsilon[}(y-K) \\
&+ \mathbf{1}_{[\varepsilon,+\infty)}(y-K),
\end{aligned}$$

and obtain the following graphs by Monte-Carlo simulation:



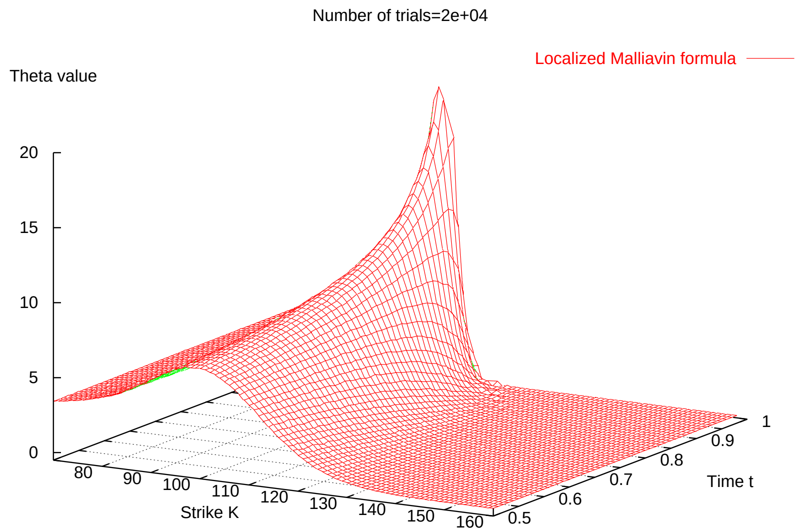
$T = 1$, $x = 100$, $\sigma = 0.15$, $r = 0.05$, $\varepsilon = 0.001$.

Figure 8 : Theta as a function of t and K by the localized Malliavin method

In the case of European options we take

$$h_\varepsilon(y) = \frac{1}{4\varepsilon}(y - (K - \varepsilon))^2 \mathbf{1}_{[-\varepsilon,\varepsilon]}(y - K) + (y - K)\mathbf{1}_{]\varepsilon,+\infty)}(y - K),$$

which yields the following graph by Monte Carlo simulation:



T=1, x=100, σ=0.15, r=0.05, ε=0.001

Figure 9 : Theta as a function of t and K by the localized Malliavin method

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